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WEAK IDENTITIES, PHANTOM MAPS AND *H*-SPACES

BY

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ABSTRACT

For a pointed space X, let Aut(X) be the group of pointed homotopy classes of pointed self-homotopy equivalences of X and let WI(X) be the normal subgroup of Aut(X) consisting of weak identities, that is, elements represented by maps weakly homotopic to the identity map. If X is a path-connected CW-space satisfying certain finiteness conditions, then the author has shown elsewhere that the quotient group Aut(X)/WI(X) is a residually finite group. If, in addition, X supports a homotopy-associative H-space structure, has a finite fundamental group, and has finitely generated higher homotopy groups, it is shown here that any normal subgroup N of Aut(X) rendering the quotient group Aut(X)/N residually finite must contain WI(X). The proof relies on establishing an isomorphism between WI(X) and the group Ph(X) of pointed homotopy classes of phantom self-maps of X and making a detailed analysis of the group-theoretic structure of the latter, following W. Meier and A. Zabrodsky.

A few years ago, I became interested in the group-theoretic behavior of the "automorphism group" Aut(X) of a certain kind of (possibly infinite-dimensional) CW-space X. When X is, in addition, a homotopy-associative H-space, I noticed that the set of homotopy classes of phantom self-maps of X, Ph(X), entered naturally into the picture and I initiated (in late 1983) a correspondence with Alex Zabrodsky, hoping he could tell me something about the structure of Ph(X).

By sheer coincidence, Zabrodsky had been working on refinements and consequences of Haynes Miller's theorem proving the Sullivan conjecture

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and had derived considerable information on the structure of $\mathrm{Ph}(X, Y)$, the set of homotopy classes of phantom maps from X to Y, under suitable restrictions on X and Y. As a result of our correspondence, it became possible to deduce a highly satisfactory supplement to a theorem I had discovered asserting the residual finiteness of a certain quotient group of $\mathrm{Aut}(X)$.

At that time, it was my hope that we would eventually produce a joint note on the results obtained. Unfortunately, as often happens, other matters took precedence, time passed and the suggestion to write up a joint paper was never made (by either one of us). The present work is my development of the essential points of our brief collaboration and will hopefully serve as a fitting tribute to a most talented topologist and fine individual.

§1. Phantom maps: generalities

Throughout, all spaces will be assumed to be pointed and path-connected CW-spaces; maps between spaces will be assumed to be pointed. The set of (pointed) homotopy classes of maps from X to Y is written, as usual, as $[X, Y]$.

For u in $[X, Y]$, $\mathrm{Ph}(X, Y; u)$ is defined as the set of all v in $[X, Y]$ such that v is "weakly homotopic" to u; that is, if W is any *finite* complex and $f\colon W \to X$ any map, then $u \cdot f = v \cdot f$. [Here, and elsewhere, the distinction between a map and its homotopy class will be blurred whenever it is deemed safe to do so.] If $u = 0$, the constant map taking X to the basepoint of Y, then $\mathrm{Ph}(X, Y; u)$ is written $\mathrm{Ph}(X, Y)$ and referred to as the set of (homotopy classes of) phantom maps from X to Y. [See [M1], [M2], [M3], [Z2], where the notation $\theta(X, Y)$ is used instead of $\mathrm{Ph}(X, Y)$. Earlier work on the subject, using a somewhat different notion of phantom map, may be found in [AW], [G].]

In general, one cannot expect a relationship among the various sets $\mathrm{Ph}(X, Y; u)$, u in $[X, Y]$. There is one situation, however, in which there is a close relationship among the $\mathrm{Ph}(X, Y; u)$. Namely, suppose that Y admits a homotopy-associative H-space structure. Then the set $[X, Y]$ is naturally furnished with a group structure, which we write additively, even though it is generally non-abelian. The proof of the following is easily supplied.

PROPOSITION 1.1. (i) *With respect to the indicated group structure on* $[X, Y]$, *the set* $\mathrm{Ph}(X, Y)$ *is a normal subgroup of* $[X, Y]$.

(ii) *For u in* $[X, Y]$, $\mathrm{Ph}(X, Y; u)$ *coincides with the coset* $\mathrm{Ph}(X, Y) + u$.

$\mathrm{Ph}(X, Y; u)$ has fairly evident functorial properties, which we are content to spell out in the case $u = 0$. Thus, $f: X_1 \to X_2$ induces $f^*: [X_2, Y] \to [X_1, Y]$, which restricts to

$$(1.1) \qquad \overline{f^*}: \mathrm{Ph}(X_2, Y) \to \mathrm{Ph}(X_1, Y).$$

When Y is a homotopy-associative H-space, f^* and $\overline{f^*}$ are group homomorphisms. This may be expressed as

$$(1.2) \qquad (\alpha + \beta) . f = \alpha . f + \beta . f, \qquad \alpha, \beta \text{ in } [X_2, Y],$$

with the three terms $\alpha . f$, $\beta . f$, $(\alpha + \beta) . f$ all in $\mathrm{Ph}(X_1, Y)$ if α, β are in $\mathrm{Ph}(X_2, Y)$. Also, $g: Y_1 \to Y_2$ induces $g_*: [X, Y_1] \to [X, Y_2]$, which restricts to

$$(1.3) \qquad \overline{g_*}: \mathrm{Ph}(X, Y_1) \to \mathrm{Ph}(X, Y_2).$$

When Y_1 and Y_2 are homotopy-associative H-spaces *and* g is an H-map, g_* and $\overline{g_*}$ are group homomorphisms. This may be expressed as

$$(1.4) \qquad g . (\alpha + \beta) = g . \alpha + g . \beta, \qquad \alpha, \beta \text{ in } [X, Y_1], \quad g \text{ an } H\text{-map},$$

with the three terms $g . \alpha$, $g . \beta$, $g . (\alpha + \beta)$ all in $\mathrm{Ph}(X, Y_2)$ if α, β are in $\mathrm{Ph}(X, Y_1)$.

Let now $X = Y$. $\mathrm{Ph}(X, Y; u)$ may then be abbreviated to $\mathrm{Ph}(X; u)$, $\mathrm{Ph}(X, Y)$ to $\mathrm{Ph}(X)$. If $u = 1_X$, the identity map of X to itself, $\mathrm{Ph}(X; 1_X)$ was written as $\mathrm{WI}(X)$ in [R] and referred to as the set of weak identities. The set $[X, X]$ has a multiplicative semigroup structure arising from the composition of maps. If X is a homotopy-associative H-space, giving rise to the additive group structure on $[X, X]$ discussed above, there is a left-distributive law

$$(1.5) \qquad (\alpha + \beta) . \gamma = \alpha . \gamma + \beta . \gamma, \qquad \alpha, \beta, \gamma \text{ in } [X, X],$$

connecting these two binary operations on $[X, X]$ (see (1.2)), but the right-distributive law

$$(1.6) \qquad \gamma . (\alpha + \beta) = \gamma . \alpha + \gamma . \beta, \qquad \alpha, \beta, \gamma \text{ in } [X, X],$$

is valid only when γ is an H-map (see (1.4)).

Both $\mathrm{WI}(X)$ and $\mathrm{Ph}(X)$ are multiplicative sub-semigroups of $[X, X]$. In fact,

PROPOSITION 1.2. (i) $\mathrm{WI}(X)$ *is a normal subgroup of the multiplicative group of units* $[X, X]^*$ (*aka* $\mathrm{Aut}(X)$) *in* $[X, X]$.

(ii) $\mathrm{Ph}(X)$ *is a "two-sided ideal" of* $[X, X]$ *in the sense that, aside from* *being an additive normal subgroup of* $[X, X]$ *in the event* X *is a homotopy-associative H-space,* $\phi . \alpha$, $\alpha . \phi$ *are in* $\mathrm{Ph}(X)$ *whenever* ϕ *is in* $\mathrm{Ph}(X)$ *and* α *is in* $[X, X]$.

Observe that the group structure on $\mathrm{WI}(X)$ is independent of any H-space structure on X; in fact, it is clearly defined even when X admits no H-space structure. On the other hand, the group structure on $\mathrm{Ph}(X)$ does seem to depend on the H-space structure on X. Nevertheless, as will be seen in §3, there is, rather surprisingly, an isomorphism from the (additive) group $\mathrm{Ph}(X)$ to the (multiplicative) group $\mathrm{WI}(X)$ given by the one-to-one correspondence

$$\phi \to \phi + 1_X, \qquad \phi \text{ in } \mathrm{Ph}(X),$$

implicit in Proposition 1.1(ii), at least if additional restrictions are imposed on X. Furthermore, under these same restrictions on X, $\mathrm{Ph}(X)$ turns out to possess some remarkable group-theoretic properties (§2) enabling us to prove the main result (§3).

§2. Phantom maps: rationalization

For the remainder of the paper, X and Y will be taken to be nilpotent, of finite homotopical type (that is, $\pi_n X$ is finitely generated for $n \geqq 1$) and with finite fundamental group. We write $X_{(0)}$ for the rationalization of X, $r: X \to X_{(0)}$ for a rationalization map and X_τ for the homotopy-fibre of r. Thus we have a fibration

$$X_\tau \xrightarrow{i} X \xrightarrow{r} X_{(0)}. \tag{2.1}$$

Since $\pi_1 X$ is finite, $\pi_1 X_{(0)} = 0$ and so X_τ is path-connected. According to [HMR; Th. II.2.2], X_τ is therefore itself a nilpotent space. The homotopy exact sequence associated to (2.1) gives rise to short exact sequences

$$Q/Z + \cdots + Q/Z \rightarrowtail \pi_n X_\tau \twoheadrightarrow \tau(\pi_n X), \qquad n \geqq 1,$$

$\tau(\pi_n X)$ denoting the (finite) torsion subgroup of $\pi_n X$; thus the groups $\pi_n X_\tau$, $n \geqq 1$ are *locally finite* (countable) groups.

We will need the following two general lemmas concerning rationalization.

Lemma 2.1. *The fibration* (2.1) *is also, up to homotopy, a cofibration.*

A yet more general result along these lines may be found in [A], where references to earlier work of a similar nature due to R. J. Milgram, Z. Wojtkowiak and S. J. Schiffman are given. It may be remarked that while the condition that X have finite homotopical type is irrelevant for Lemma 2.1, the condition that $\pi_1 X$ be a torsion group is necessary, as well as sufficient.

LEMMA 2.2. *If $j: U \to V$ is a rational homotopy equivalence of nilpotent spaces of finite homotopical type, then $j_*: [X_{(0)}, U] \to [X_{(0)}, V]$ is a bijection.* (See [Z2; 1.1.5].)

For completeness, we sketch a proof of Lemma 2.2. As $X_{(0)}$ is 1-connected, we may asume $X_{(0)}$ decomposed as a "rational" cell complex ([HMR; p. 57]). Thus, $X^2_{(0)}$ is a wedge $VS^2_{(0)}$ of rational 2-spheres, $X^{n+1}_{(0)}$ is obtained from $X^n_{(0)}$ by attaching a disjoint union of "rational $(n+1)$-cells" by means of a map $VS^n_{(0)} \to X^n_{(0)}$, and $X_{(0)} = \varinjlim X^n_{(0)}$. We proceed inductively to show

$$j_*: [X^n_{(0)}, U] \to [X^n_{(0)}, V]$$

is a bijection.

Since $X_{(0)}$ is 1-connected, U and V may also be taken to be 1-connected and hence the homotopy-fibre F of j is path-connected, nilpotent. For $n = 2$, we have an exact sequence

$$\to [VS^2_{(0)}, F] \to [VS^2_{(0)}, U] \xrightarrow{j_*} [VS^2_{(0)}, V] \to [VS^1_{(0)}, F] \to .$$

Since $\pi_n F$ is finite for all $n \geqq 1$,

$$H^n(S^n_{(0)}; \pi_n F) = \mathrm{Hom}(H_n S^n_{(0)}, \pi_n F) = 0,$$

$$H^{n+1}(S^n_{(0)}; \pi_{n+1} F) = \mathrm{Ext}(H_n S^n_{(0)}, \pi_{n+1} F) = 0, \qquad n \geqq 1.$$

By obstruction theory, $[S^n_{(0)}, F] = 0$, $n \geqq 1$. Therefore the case $n = 2$ is settled. The inductive step is proved by applying a Five Lemma argument to the evident commutative diagram with exact rows

$$\begin{array}{ccccccccc} [VS^n_{(0)}, U] & \leftarrow & [X^n_{(0)}, U] & \leftarrow & [X^{n+1}_{(0)}, U] & \leftarrow & [VS^{n+1}_{(0)}, U] & \leftarrow & [\Sigma X^n_{(0)}, U] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [VS^n_{(0)}, V] & \leftarrow & [X^n_{(0)}, V] & \leftarrow & [X^{n+1}_{(0)}, V] & \leftarrow & [VS^{n+1}_{(0)}, V] & \leftarrow & [\Sigma X^n_{(0)}, V]. \end{array}$$

A simple limit argument then establishes that

$$j_*: [X_{(0)}, U] \to [X_{(0)}, V]$$

is a bijection.

We turn now to a closer examination of the set $\mathrm{Ph}(X, Y)$. The following result of Meier and Zabrodsky is crucial ([Z2; 2.1.2.]).

PROPOSITION 2.1. *ϕ is in* $\mathrm{Ph}(X, Y)$ *iff there exists $\tilde{\phi}$ in $[X_{(0)}, Y]$ such that*

$$\phi = \tilde{\phi} . r.$$

By virtue of Lemma 2.1, there is an exact sequence of sets

$$[X_\tau, Y] \xleftarrow{i^*} [X, Y] \xleftarrow{r^*} [X_{(0)}, Y] \xleftarrow{q^*} [\Sigma X_\tau, Y] \longleftarrow, \tag{2.2}$$

$q: X_{(0)} \to \Sigma X_\tau$ being the "connecting map" in the fibration-cofibration (2.1). Proposition 2.1 identifies $\mathrm{Ph}(X, Y)$ with the image of r^* in (2.2). We study the latter when Y is an H-space.

Being an H-space, Y is rationally equivalent to a product of Eilenberg–MacLane spaces $K(Z, n)$. More precisely, let

$$K = \prod K(\pi_n' Y, n),$$

where $\pi_n' Y$ denotes the quotient group $\pi_n Y / \tau(\pi_n Y)$. Then there is a rational homotopy equivalence $j: Y \to K$. Furthermore, j may be chosen to be an H-map for the given H-space structure on Y and for some (not necessarily unique) H-space structure on K — see [Z1; 4.4.3]. If

$$K_{(0)} = \prod K(\pi_n' Y \otimes Q, n)$$

and $\rho: K \to K_{(0)}$ is the evident (rationalization) map, then the composition $\rho . j$ serves as a rationalization map $r: Y \to Y_{(0)}$, which will be used later.

Now, (2.2) embeds as the top row in a commutative diagram

$$\begin{array}{ccccccc} [X_\tau, Y] & \longleftarrow & [X, Y] & \xleftarrow{r^*} & [X_{(0)}, Y] & \xleftarrow{q^*} & [\Sigma X_\tau, Y] \\ \downarrow & & \downarrow & & \downarrow j_* & & \downarrow j_* \\ [X_\tau, K] & \longleftarrow & [X, K] & \xleftarrow{r^*} & [X_{(0)}, K] & \xleftarrow{q^*} & [\Sigma X_\tau, K]. \end{array} \tag{2.3}$$

With respect to the given H-space structure on Y and the H-space structure "induced" on K, all the maps in (2.3) are homomorphisms of algebraic loops (for a definition, see [B]). We wish to show that some of these algebraic loops are actually groups, and for this purpose, we need a lemma.

LEMMA 2.3. *$r^*: [X_{(0)}, K] \to [X, K]$ is the trivial map.*

PROOF. As K is a product of $K(Z, n)$'s it suffices to prove that

$$r^*: H^n(X_{(0)}; Z) \to H^n(X; Z)$$

is the zero map, $n > 0$. By the Universal Coefficient Theorem, there is a commutative diagram with short exact rows

$$\begin{array}{ccccc} Ext(H_{n-1}X_{(0)}, Z) & \rightarrowtail & H^n(X_{(0)}; Z) & \twoheadrightarrow & \mathrm{Hom}(H_nX_{(0)}, Z) \\ \downarrow & & \downarrow r^* & & \downarrow \\ \mathrm{Ext}(H_{n-1}X, Z) & \rightarrowtail & H^n(X; Z) & \twoheadrightarrow & \mathrm{Hom}(H_nX, Z). \end{array} \tag{2.4}$$

As X is of finite type, $\mathrm{Ext}(H_{n-1}X_{(0)}, Z)$ is divisible while $\mathrm{Ext}(H_{n-1}X, Z)$ is finite; also, $\mathrm{Hom}(H_nX_{(0)}, Z) = 0$. Thus r^*, in (2.4), is indeed the zero map.

It follows from Lemma 2.3 that $q^*: [\Sigma X_\tau, K] \to [X_{(0)}, K]$, in (2.3), is surjective. But, ΣX_τ being a suspension, the algebraic loop structure on $[\Sigma X_\tau, K]$ is a group structure. In fact, this group structure is independent of the H-space structure on K and is abelian. Hence, the algebraic loop structure on $[X_{(0)}, K]$ is also an (abelian) group structure, independent of the H-space structure of K.

The map $j_*: [X_{(0)}, Y] \to [X_{(0)}, K]$, in (2.3), is both a bijection (Lemma 2.2) and a homomorphism of algebraic loops. Thus, the algebraic loop structure on $[X_{(0)}, Y]$ is actually an (abelian) group structure, independent of the H-space structure on Y (whether homotopy-associative or not), and $j_*: [X_{(0)}, Y] \to [X_{(0)}, K]$ is an isomorphism of groups. We may now state

THEOREM 2.1. *Let Y be an H-space. The natural algebraic loop structure on* $\mathrm{Ph}(X, Y)$ (*compare Proposition* 1.1(i)) *is a group structure which is abelian, independent of the H-space structure on Y and divisible* (compare [Z2; 2.1.4]).

PROOF. We have already noted (Proposition 2.1 *et seq.*) the identification

$$\mathrm{Ph}(X, Y) \cong r^*[X_{(0)}, Y].$$

Chasing around the diagram (2.3), we find

$$r^*[X_{(0)}, Y] \cong [X_{(0)}, Y]/q^*[\Sigma X_\tau, Y] \tag{2.5}$$

$$\cong [X_{(0)}, K]/q^*j_*[\Sigma X_\tau, Y]. \tag{2.6}$$

All the assertions of Theorem 2.1, except for the divisibility, follow from (2.5). To obtain the divisibility statement, impose the standard H-space structure on K and identify

$$[X_{(0)}, K] \cong \prod H^n(X_{(0)}; \pi_n' Y)$$
$$\cong \prod \operatorname{Ext}(H_{n-1}X_{(0)}, \pi_n' Y).$$

The latter group, and hence any quotient group thereof, is certainly divisible, so (2.6) completes the proof.

Imposing further conditions on X, Y leads to more precise information on $\operatorname{Ph}(X, Y)$.

THEOREM 2.2 ([Z2; 4.1.]). *Suppose that X has only finitely many non-zero homotopy groups and that (the H-space) Y has the homotopy type of a finite complex. Then*

(i) $\operatorname{Ph}(X, Y) \cong \Pi \operatorname{Ext}(H_{n-1}X_{(0)}, \pi_n' Y)$;

(ii) *If there is an integer n for which both $H_n X$ and $\pi_{n+1} Y$ have Z-rank $\geqq 1$, then* $\operatorname{Ph}(X, Y)$ *is uncountable. If no such integer n exists, then* $\operatorname{Ph}(X, Y) = 0$ (compare [M2; Prop. 2]).

PROOF. For part (i), it suffices, by the proof of Theorem 2.1 (see (2.6)), to show that $[\Sigma X_\tau, Y] = 0$, given the additional conditions on X, Y. But this follows from Zabrodsky's extension of Haynes Miller's theorem, as explained in [Z2; 3.2].

Part (ii) follows readily from part (i), using the calculation

$$\operatorname{Ext}(Q, Z) \cong R.$$

REMARK. It is interesting to observe that Theorem 2.2 falls completely apart if the assumption on the finiteness of $\pi_1 X$ is removed. For instance, if $X = S^1 \times S^1$ and $Y = S^3$, then $\operatorname{Ph}(X, Y) = 0$ while $\Pi \operatorname{Ext}(H_{n-1}X_{(0)}, \pi_n' Y) \cong \mathbf{R}$.

§3. Phantom maps and weak identities

In this section, we take $X = Y$ and require X to be a homotopy-associative H-space. Recall (Proposition 1.1(ii)) the bijection $\operatorname{Ph}(X) \to \operatorname{WI}(X)$ given by

$$\phi \to \phi + 1_X,$$

where " + " is the binary operation induced by the H-space structure on X.

THEOREM 3.1. *The bijection* $\operatorname{Ph}(X) \to \operatorname{WI}(X)$ *just described is an isomorphism of groups.*

PROOF. For ϕ_1, ϕ_2 in $\operatorname{Ph}(X)$, we must compute the product $(\phi_1 + 1) . (\phi_2 + 1)$. (We abbreviate 1_X to 1.) First use Proposition 2.1 to write

$$\phi_i = \bar{\phi}_i \,.\, r, \qquad i = 1, 2.$$

Recall, from §2, that r may be chosen to be of the form

$$X \xrightarrow{j} K \xrightarrow{\rho} K_{(0)} = X_{(0)}.$$

Now,

$$\begin{aligned}
(\phi_1 + 1)\,.\,(\phi_2 + 1) &= \phi_1\,.\,(\phi_2 + 1) + (\phi_2 + 1) \qquad \text{(see (1.5))} \\
&= (\tilde{\phi}_1\,.\,r)\,.\,(\tilde{\phi}_2\,.\,r + 1) + (\phi_2 + 1) \\
&= \tilde{\phi}_1\,.\,\{r\,.\,(\tilde{\phi}_2\,.\,r + 1)\} + (\phi_2 + 1) \\
&= \tilde{\phi}_1\,.\,\{r\,.\,\tilde{\phi}_2\,.\,r + r\} + (\phi_2 + 1) \qquad \text{(see (1.6); } r \text{ is an } H\text{-map)} \\
&= \tilde{\phi}_1\,.\,\{\rho\,.\,j\,.\,\tilde{\phi}_2\,.\,r + r\} + (\phi_2 + 1) \\
&= \tilde{\phi}_1\,.\,\{0 + r\} + (\phi_2 + 1) \qquad (j\,.\,\tilde{\phi}_2\,.\,r = 0 \text{ by Lemma 2.3)} \\
&= \phi_1 + \phi_2 + 1,
\end{aligned}$$

and Theorem 3.1 is proved.

REMARKS. (1) The calculations in Theorem 3.1 take place in $[X, X]$, not in $\mathrm{Ph}(X)$. The latter is a group even if the H-space structure on X is not homotopy-associative but the former will not be a group in general unless X is assumed homotopy-associative.

(2) A portion of the above calculation shows that $\mathrm{Ph}(X)$ is nilpotent of class 2 in the sense that the product of any 2 elements in $\mathrm{Ph}(X)$ is 0. The proof depends on being able to factor r as $\rho\,.\,j$ with the target of j being a product of $K(Z, n)$'s. Hence, this nilpotency result remains valid even if X is only assumed to be a "rational H-space." Does it continue to hold even more generally?

Returning to the situation in Theorem 3.1, we see that $\phi^n = 0$, $n \geqq 2$, for any ϕ in $\mathrm{Ph}(X)$. Thus the formal exponential expression $\exp(\phi) = \Sigma\phi^n/n!$ reduces to $1 + \phi$ (which may differ from $\phi + 1$ since $[X, X]$, unlike $\mathrm{Ph}(X)$, need not be abelian). The isomorphism in Theorem 3.1 may therefore be described as an "anti-exponential" isomorphism.

Theorem 3.1 leads to an interesting supplement to [R; Th. 4.4], which asserts that, under rather general finiteness conditions on X (all nilpotent spaces of finite homotopical type are included), the quotient group $\mathrm{Aut}(X)/\mathrm{WI}(X)$ is a residually finite group.

THEOREM 3.2. *If X is a homotopy-associative H-space of finite homotopical type and with finite fundamental group, and if N is any normal subgroup of* $\mathrm{Aut}(X)$ *yielding a residually finite quotient group* $\mathrm{Aut}(X)/N$, *then* $N \supset \mathrm{WI}(X)$.

PROOF. Virtually by the definition of residual finiteness, the group $\mathrm{Aut}(X)/N$ embeds into a Cartesian product of finite groups $\Pi\, \Gamma_\alpha$. Consider the composition

$$(3.1) \qquad \mathrm{WI}(X) \xhookrightarrow{\text{inclusion}} \mathrm{Aut}(X) \xrightarrow{\text{projection}} \mathrm{Aut}(X)/N \xhookrightarrow{\text{embedding}} \Pi\, \Gamma_\alpha .$$

By Theorems 2.1 and 3.1, $\mathrm{WI}(X)$ is (abelian) divisible. Therefore, each Γ_α being finite, (3.1) is the trivial map and so $\mathrm{WI}(X) \subset N$.

QUESTION. Does Theorem 3.2 continue to hold for more general X?

We conclude by giving an example of a space T of the type described in Theorem 3.2 in which $\mathrm{WI}(T)$ is uncountable. First observe that a map $f: X \to Y$ induces a self-map $X \times Y \to X \times Y$ defined by

$$(x, y) \to (0, f(x)).$$

There results an injective map

$$[X, Y] \to [X \times Y, X \times Y]$$

which, in fact, restricts to an (injective) map

$$\mathrm{Ph}(X, Y) \to \mathrm{Ph}(X \times Y).$$

Take $X = K(Z, 2) = CP^\infty$, $Y = S^3$. These two spaces satisfy the conditions of Theorem 2.2 and are both homotopy-associative H-spaces (using the standard H-space structures). Furthermore,

$$H_2 X \cong Z \cong \pi_3 Y.$$

By Theorem 2.2, $\mathrm{Ph}(X, Y)$ is uncountable (compare [G], [M2], [M3]), hence also $\mathrm{Ph}(X \times Y)$ is uncountable. It follows that $\mathrm{WI}(X \times Y)$ is itself uncountable; $X \times Y$ is the promised space T.

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